Approximation and Interpolation

by

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Introduction.

The main purpose with this paper is to apply the techniques developed by A.G. Vitushkin in [8] to problems concerning some spaces of analytic functions.

Let $X \in \mathbb{C}$ be compact, $U = X^o$ its interior. Let $B \subset \partial X$ be open relative to ∂X. $H_B^\infty(X^o)$ consists of all bounded continuous functions on $X^o \cup B$ being analytic in X^o. We are interested in when every bounded analytic function in U can be uniformly approximated on the sets $F \subset X^o$ closed relative to $X^o \cup B$ by functions in $H_B^\infty(X\ )$.

The paper is divided into three sections. Section 1 is intended only as a motivation for the problems to be studied in the next sections and we there solve the approximation problem mentioned above in case X is the closed unit-disc. The proof is based on the theory of H^p-spaces.

In section 2 we apply the techniques developed by Vitushkin to generalize the result. We here make use of a theorem proved recently by T.W. Gamelin and J. Garnett. We show that whenever the approximation problem can be solved when $B = \partial X$, then it can be solved for any subset B open relative to ∂X. In section 2 we also generalize a theorem proved by E. A. Heard and J.H. Wells in case $X = \{z: |z| \leq 1\}$ concerning interpolationsets for $H_B^\infty(X^o)$.

In section 3 we study the following problem :

Let $\{Z_n\}$ be a sequence in an open set $U \subset \mathbb{C}$. Suppose that for every bounded sequence $\{W_n\} \subset \mathbb{C}$ we can find a bounded analytic function f such that $f(Z_n) = W_n$ for $n = 1,2,\cdots$. Is it possible to find an open set $0 \supset \overline{U} \setminus E$ (where E is the clusterpoints of $\{Z_n\}$ on ∂U) so that the interpolation problem mentioned above can be solved by bounded analytic functions defined in 0 ?

Problems of this kind were first studied by Akutowiz and Carleson. In case $U = D = \{z: |z| < 1\}$ Heard and Wells extended one of the results they proved. Later J. Detraz proved an interesting theorem generalizing the result of Heard and Wells, but still for the case $U = D$. In section 3 we prove that the problem mentioned above has a positive solution for a large class of open subsets U of $\mathbb{C}$.

Notation.

In the following X is a compact subset of $\mathbb{C}$ and $A(X)$ consists of all continuous functions on X being analytic in X^o. $R(X)$ is all functions in $A(X)$ being uniform limits on X of rational functions with poles outside X.

$H^\infty(0)$ is defined whenever $0 \subset \mathbb{C}$ is open as all bounded analytic functions on 0. We say that $A(X)$ is pointwise boundedly dense in $H^\infty(X^o)$ if every $f \in H^\infty(X^o)$ is a pointwise limit of a bounded sequence of functions in $A(X)$. Whenever S is a topological space $C(S)$ is the Banach-algebra of all bounded continuous complexvalued functions on S.

If $S \subset \mathbb{C}$, it has the topology induced from $\mathbb{C}$.

We assume the reader knows the definitions of analytic capacity and continuous analytic capacity and the basic results from the theory of analytic capacity and rational approximation.

A convenient reference is Ch. VIII of [4]. If $E \subset \mathbb{C}$ then $\gamma(E)$, $\alpha(E)$ denotes the analytic capacity and the continuous analytic capacity respectively.

If B is a subset of ∂X (the boundary of X) we define $H_B^\infty(X^o)$ $\{f \in C(X^o \cup B) : f|_{X^o} \in H^\infty(X^o)\}$

It is a Banach algebra under the usual sup norm.

Finally if $\delta > 0$, $z_o \in \mathbb{C}$ then $\Delta(z_o,\delta) = \{z \in \mathbb{C} \mid |z-z_o| < \delta\}$. If f is bounded and measurable on $\mathbb{C}$ we put $\|f\| = \|f\|_\infty$ where $\|\ \|_\infty$ is the essential supremum of $|f|$ with respect to plane Lebesque-measure.

Section 1.

Let $D = \{z : |z| < 1\}$ and $T = \partial D$ be the circle-group. If u is a real integrable function on T we define the analytic function $H_u(z)$ by

$$H_u(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\, u(\theta)d\theta \qquad\qquad z \in D.$$

It is well known (see [7] p. 67) that we can factor an $f \in H^\infty(D)$, into $f = f_1 f_2$ where $f_1, f_2 \in H^\infty(D)$, $\lim |f_1(r\, e^{i\theta})| = 1$ a.e. on T and $f_2 = \lambda \exp(H_u)$ where $|\lambda| = 1$ and H_u is as above. f_1 is called an inner function, f_2 an outer function.

A Bascke-product is an inner function given by a product

$$B(z) = \lambda \cdot Z^k \cdot \prod_n \frac{|\alpha_n|}{\alpha_n}\ \frac{\alpha_n - z}{1-\overline{\alpha}_n z}$$

and the convergence is uniform on compact subsets of $\mathbb{C}$ at a positive distance from the set $\{\frac{1}{\overline{\alpha}_n} \mid n = 1,2,\cdots\}$. Let $E \subset T$ be compact and $B = T \setminus E$. Using the notation above we now have :

Theorem 1.1.

Suppose $F \subset D$ is closed relative to $D \cup B$.

Given $h \in H^{\infty}(D)$ and $\varepsilon > 0$ there exists $f \in H^{\infty}_{B}$ such that $\|h-f\|_{F} < \varepsilon$ and $\|f\| \leq \|h\|$. If h is an inner (outer) function we can choose f to be inner (outer).

Theorem 1 follows from the factorization Theorem mentioned above and the following three lemmata:

Lemma 1.1

Every inner function $f \in H^{\infty}_{B}$ is an uniform limit of Blascke-product in H^{∞}_{B}.

Lemma 1.2

Given $\varepsilon > 0$ and a Blascke-product $G \in H^{\infty}(D)$ there exists a Blascke-product $G' \in H^{\infty}_{B}$ such that $\|G - G'\|_{F} < \varepsilon$.

Lemma 1.3

Given $u \leq 0$ in $L^{1}(T)$ and $\varepsilon > 0$ there exsists an outer function $G \in H^{\infty}_{B}$ such that $\|G - \exp H_{(u)}\|_{F} < \varepsilon$ and $\|G\| \leq 1$.

Lemma 1.1 is proved at page 176 in [7].

Proof of Lemma 1.2

Let $B = \bigcup\limits_{1}^{\infty} J_{n}$ where the union is disjoint and each J_{n} is a half-open arc such that every compact subset K of B is covered by a finite number of the arcs. Let $D_{n} = \{z \in D\setminus\{0\} : \frac{z}{|z|} \in J_{n}\}$

From G we take a subproduct G_1 having only a finite number of factors with zeros in D_1 and such that

$$\|G - G_1\|_{F} < \frac{\varepsilon}{2^2}$$

Then we take a subproduct G_2 of G_1 having only a finite number of factors with zeros in D_2 and such that $\|G_1-G_2\| < \frac{\varepsilon}{2^3}$.

We proceed in this way and get a sequence of Blascke-products G_n.

A subsequence G_{n_k} converges uniformly on compact subsets of an open set O containing $D \bigcup T \setminus E$ to an analytic function whose restriction G_o to D is an inner function in H^∞_B with $\|B-G_o\|_F \leq \frac{\varepsilon}{2}$. Using lemma 1 we now have lemma 1.2.

Proof of lemma 1.3

Let $u_1 = u|_B$ and $u_2 = u - u_1$.

Choose a realvalued function $v \in L'(T)$ continuous differentiable on B such that $v \leq 0$, $v = 0$ on E and such that $\sup\limits_{z \in F} |H_{(u_1)}(z) - H_{(v)}(z)| < \frac{\varepsilon}{e}$.

Then the function $G = \exp\left[H_{(v+u_2)}\right]$ is the required one.

Let now $C(F)$ be the Banach-space of all bounded continuous functions on F with $\|f\| = \sup\{|f_{(z)}|;\ z \in F\}$. Then we have:

Corollary 1.1

If $L = H^\infty|F$ is equal to $C(F)$ then $B_E|_F = H^\infty|_F$.

Corollary 1.1 is an immediate consequence of Theorem 1.1 and the following lemma that will be useful to us several times :

Lemma 1.4

Suppose $T: X \to Y$ is a linear continuous map from a Banach-space X into a Banach-space Y and there exist numbers $t \in \langle 0,1\rangle$ and $M < \infty$ such that for every $y \in Y$ with $\|y\| \leq 1$ there exist $x \in X$ such that $\|y - Tx\| < t$ and $\|x\| \leq M$.

Then $TX = Y$ and if $y \in Y$ then $y = Tx$ for some $x \in X$ with $\|x\| \leq \frac{M}{1-t}$

Proof.

Let $y \in Y$ and $\|y\| = 1$. Choose a sequence $\{x_n\} \subset X$ such that $\|x_n\| \leq Mt^n$, $n = 1,2,\cdots$ and such that $\|T(\sum_1^m x_n) - y\| \leq t^m$ for $m = 1,2,\cdots$. Then $x = \sum_1^\infty x_n \in X$, $\|x\| \leq \frac{M}{1-t}$ and $Tx = y$.

The corollary follows now by letting

$$X = H_B^\infty \qquad \text{and} \qquad Y = H^\infty|_{F \cap D}$$

and $T: X \to Y$ be the restriction-map.

Section 2

We now generalize theorem 1.1.

Theorem 2.1

Suppose $A(X)$ is pointwise boundedly dense in $H^\infty(X^o)$.

Then there exists a constant k such that if $B \subset \partial X$ is open relative to ∂X, $h \in H^\infty(X^o)$, $F \subset X^o$ is closed relative to $X^o \cup B$ and $\varepsilon > 0$ we can find a function f $H_B^\infty(X^o)$ such that $\|f\| \leq k\|h\|$ and $\|f-h\|_F < \varepsilon$.

Proof.

From the hypothesis we get that there exist constants c and r such that

(1) $$\gamma(\Delta(z,\delta) \setminus X^o) \leq c\alpha(\Delta(z,r\delta) \setminus X^o)$$

whenever $z \in \mathbb{C}$, $\delta > 0$.

That (1) in fact is equivalent with the hypothesis of the theorem follows from theorem 2.2 in [5].

Choose for a fixed $S > 0$ and $k = 1,2,\cdots$, points $z_{k\delta}$ and functions $\phi_{k\delta}$: $\Delta(z_{k\delta},\delta) \to [0,1]$ such that

i) $\phi_{k\delta} \in C_0^1\, \Delta(z_{k\delta},\delta)$

ii) $\sum_1^\infty \phi_{k\delta} \equiv 1$ in $\mathbb{C}$

iii) $\left\|\frac{\partial\phi_{k\delta}}{\partial\bar{z}}\right\| \leq \frac{4}{\delta}$

iv) No complex number is contained in more than 25 of the discs $\Delta_k = (z_{k\delta},\delta)$ (See more about this construction in Ch. VIII in [4]). If f is a bounded measurable function on $\mathbb{C}$ and $\phi \in C_0^1(\mathbb{C})$ we define

(2) $$T\phi f(\zeta) = \frac{1}{\pi}\iint \frac{f(z)-f(\zeta)}{z-\zeta}\,\frac{\partial\phi}{\partial\bar{z}}(z)\,dxdy(z)$$

We have by Stokes theorem :

(3) $$T\phi f(\zeta) = f(\zeta)\cdot\phi(\zeta) + \frac{1}{\pi}\iint \frac{f(z)}{z-\zeta}\,\frac{\partial\phi}{\partial\bar{z}}(z)\,dxdy(z)$$

By (2) $T\phi f$ is analytic wherever f is analytic and by (3) $T\phi f$ is continuous wherever f is continuous.

We also have (4) $\|T\phi f\|_\infty \leq 4 \operatorname{diam} X\cdot \left\|\frac{\partial\phi}{\partial\bar{z}}\right\|_\infty \|f|_X\|_\infty$ where X is the support of ϕ. Let us also remark that $f - T\phi f$ is analytic in the interior of the set $\{z: \phi(z) = 1\}$ and that

$$T\phi f'(\infty) = \frac{-1}{\pi}\iint f(z)\,\frac{\partial\phi}{\partial\bar{z}}(z)\,dxdy(z)$$

which follows from (3) above.

We now prove Theorem 2.1.

Let f be a bounded measurable function on $\mathbb{C}$ such that $f|_{X^o} \in H^\infty(X^o)$.

Assume f analytic at (∞) and $f(\infty) = 0$

$G_k = T\phi_k f$ satisfies

(5) $\|G_k\| \leq 32 \|f\|$.

Suppose $K \subset B$ is compact and $I = \{k: \Delta_k \cap K \neq \phi\}$. Exactly as in [5] (see p.193) we can find functions $H_k \in C(S^2)$ analytic outside $\Delta(z_k, r\delta) \setminus X^o$ such that $\|H_k\| \leq b \|f\|$ where v depends only on c and r but not on δ and such that $G_k - H_k$ has a triple zero at ∞.

Let $V \supset K$ be open and $1 > \varepsilon > 0$.

If we put $g = \sum_I (G_k - H_k)$ then there exists a constant k_1 depending only on c and r such that for small δ

i) $\|g\| \leq k_1 \|f\|_V$ (k_1 independent of δ)

ii) $\|g\|_{\mathbb{C} \setminus V} < \varepsilon$.

Also g is analytic in X^o, and $f - g$ is continuous in a neighbourhood of K.

The proof of this is almost a repetition of what Gamelin and Garnett show at p.192 and 193 in [5]. We therefore omit the details. (see section 3 in this paper for more details on this problem).

Now let $B = \bigcup_1^\infty K_n$ where each K_n is compact and let $V_n \supset K_n$ be open and $(V_n \cap V_m \neq \phi) \Rightarrow |n-m| \leq 1$.

Let $F \subset X^o$ be closed relative to $X^o \cup B$ and suppose $h \in H^\infty X^o$ and $\|h\| = 1$. Put $h = 0$ outside X^o.

We also assume $V_n \cap F = \phi$ for $n = 1,2,\cdots$.

Now choose by the technique above functions $f_n \in C(S^2)$ such that

$$\|f_n\| \leq k_1 \qquad \text{and} \qquad |f_n|_{\mathbb{C}\setminus V_n} < \frac{\varepsilon}{2^{n+1}}$$

and f_n analytic in X^o and $h - f_n$ continuous near K_n.

Put $g = h - \sum_{n=1}^{\infty} f_{2n-1}$

Then $\|g\| \leq k_1 + \varepsilon \leq k_1 + 1$

and $\|g - h\|_F < \varepsilon$ since $V_n \cap F = \phi$ for every n.

Then we modify g in the same way on $K_2, K_4,$ ---- by functions $g_2, g_4,$ ----

and put $f = g - \sum_1^\infty g_{2n}$.

Then $\|f\| \leq (k_1 + 1)^2$ and $\|f - h\|_F < 2\varepsilon$.

Moreover $f \in H_B^\infty(X^o)$. Since k_1 depends only on c and r the theorem is proved.

The next result is a general version of theorem 1 proved in |6|.

<u>Theorem 2.2.</u>

Let X, B be as above and assume the hypothesis of theorem 2.1. Suppose S is a subset of $X^o \cup B$ closed in the relative topology on $X^o \cup B$.

If each compact subset of $S \cap B$ is a peak-interpolation set for $A(X)$ and $H^{\infty}(X^{o})|S \cap X^{o}$ is closed in $\mathbb{C}(S \cap X^{o})$, then $L = H_B^{\infty}(X^{o})|_S$ is closed in $\mathbb{C}(S)$ and every $f \in C(S \cap B)$ has an extension to an element of L. If $S \subset B$ then S is a peak-interpolation set for $H_B^{\infty}(X^{o})$.

To prove Theorem 3.2 we need to generalize lemma 3 of [6]. The next lemma is stated for the algebra H_B^{∞} but the result is valid in the setting of a general sup-norm algebra defined on a compact Hausdorff space. (With $H_B^{\infty}(X^{o})$ replaced by an algebra of functions defined in a natural way).

Lemma 2.1.

Suppose $K \subset B$ is closed relative to B and every compact subset of K is a peak-interpolation set for $A(X)$.

Then for every $g \in C(K)$ we can find $f \in H_B^{\infty}(X^{o})$ such that $f|K = g$ and $|f| < \|g\|$ on $X^{o} \cup B \setminus K$.

Proof.

Put $Y = X^{o} \cup B$.

We write $K = \bigcup_{1}^{\infty} K_n$ where each K_n is compact and choose open sets $V_n \supset K_n$ such that $|n-m| \leq 1$ if $V_n \cap V_m \neq \phi$.

Moreover $F \cap V_n \neq \phi$ for only finitely many n if $F \subset Y$ is compact.

Let $t \in \langle 0,1 \rangle$. Suppose $g \in C(K)$ and $\|g\| = 1$. Let $K_o = \phi$ and define $f_o \equiv 0$ on X. Let $g_k = g|K_k$.

Using that every K_n is a peak-interpolation set for A and Urysohns lemma it is possible to construct a sequence $\{f_n\}_1^{\infty}$ of functions in $A(X)$ such that for $k = 1,2,\cdots$ we have

(1)k) $\quad f_k|K_k = g_k - f_{k-1}|K_k$

(2)k) $\quad \|f_k\| = \|g_k - f_{k-1}\|K_k \le \|g\|$

(3)k) $\quad |f_k| < t\cdot 2^{-k-1}$ on $(X\setminus V_k)\cup K_{k-1}$

(4)k) $\quad \|f_k - g_k\|\,K_{k+1} < \|g\| + t\cdot 2^{-k}$

(5)k) $\quad f_k = 0$ on $(K_{k-1}\cap K_k)\cup(K_{k+1}\cap K_{k+2})$

Put $F_n = \sum_1^n f_k$. Then by (2.k) and (3.k) for $1 \le k \le n$ we get $|F_n(x)| \le 2(\|g\| + t) + t \le 2\|g\| + 3t$ for $x\in X$ and $n = 1,2,\cdots$.

If we apply (3.k) for $k = 1,2,\cdots$ we see that F_n is a uniform Cauchy sequence on compact subsets of Y and has a limit F in H_B^∞ such that $\|F\| \le 2\|g\| + 3t \le 5\|g\|$.

Suppose now that $x\in K$. Then $X\in K_n$ for some n. By (1,n) $f_{n-1}(x) + f_n(x) - g(x) = 0$ and by (3.k) for $k = 1,2,\cdots$ $|F(x) - g(x)| < \sum_{n=1}^{\infty}(t\cdot 2^{-n-1}) = t$.

By lemma 1.4 every g is equal to $f|K$ for some $f\in H_B^\infty$ with $\|f\| \le \frac{5}{1-t} \le 6$ if t is small.

Having established this partial result we look at the proof and see that it shows the following :

Lemma 2.2.

Given $\varepsilon > 0$ and subset $F\subset Y$ closed in Y for which $F\cap K = \phi$.

Then there exists a function f in H_B^∞ such that $f\equiv 1$ on K, $|f| < \varepsilon$ on F and $\|f\| \le 6$.

Proof:

Assume in the proof above that $g\equiv 1$ on K, and $V_n\cap F = \phi$ V_n, and choose the functions small on F.

From lemma 2.2 and the fact that if $g \in C(K)$ $g = f|K$ with $f \in H_B^\infty$ and $\|f\| \leq 6\|g\|$ we can prove lemma 2.1. In fact the rest of the proof follows from lemma 4.4, lemma 4.5 and theorem 4.6 in [2]. We do not want to go into details here.

Now we can prove theorem 2.2 :

Let $\varepsilon > 0$ and put

$$M = \{h \in C(S) : h|_{S\cap X^o} \in H^\infty(X^o)|S\cap X^o\}$$

It is sufficient to prove that $H_B^\infty|_S = M$. Clearly $H_B^\infty|_S \subset M$. Assume $h \in M$ and $\|h\| = 1$.

Choose by lemma 2.1 $f_1 \in H_B^\infty$ such that $f_1 = h$ on $S \cap B$ and $\|f_1\| \leq 1$.

Since $H^\infty(X^o)|S\cap X^o$ is closed in $C(S\cap X^o)$ there exists by the open mapping theorem a constant k_1 independent of $h - f_1$ and $f_2^1 \in H^\infty(X^o)$ with $\|f_2^1\| \leq k_1\|h-f_1\| \leq 2k_1$ such that $f_2^1 = h - f_1$ on $S\cap X^o$. Choose by theorem 2.1 a function $f_2 \in H_B^\infty(X^o)$ with $\|f_2\| \leq k\|f_2^1\|$ and $\|f_2 - f_2^1\|_{S\cap X^o} < \varepsilon$.

Choose an open set $V \supset S\cap B$ such that $\max(|f_2|,|f_2^1|) < 2\varepsilon$ on $V \cap S \cap X^o$ and by lemma 2.1 $f_3 \in H_B^\infty(X^o)$ such that $f_3 \equiv 0$ on $S \cap B$, $\|f_3\| \leq 2$ and $|1 - f_3| < \varepsilon$ on $S \setminus V$.

Put $f = f_1 + f_2f_3$. Then $\|f\| \leq 1 + 2kk_1$ and $\|h - f\|_S < 6\varepsilon$. Choosing $\varepsilon < \frac{1}{6}$ we have that $H_B^\infty(X^o)|_S = M$ by lemma 1.4. That each $f \in C(K)$ extends to M is a consequence of lemma 2.1.

Section 3.

We assume in this section $\phi \neq U = X^{o}$ for some compact subset X of $\mathbb{C}$.

We now state the main result of this section. The theorem can be further generalized. (See the remarks after the proof).

Theorem 3.1.

Suppose S is a relatively closed subset of U and $H^{\infty}(U)|_{S}$ is a closed subspace of $C(S)$. Suppose there exsist constants c and r such that

$$(*): \quad \gamma(\Delta(z,\delta)\setminus U) \leq c\gamma(\Delta(z,r\delta)\setminus X)$$

whenever $z \in \mathbb{C}$, $\delta > 0$ and $\Delta(z,\delta)\cap \overline{S} = \phi$. Then there exsits an open set $O \supset X\setminus(\overline{S}\setminus S)$ such that $H^{\infty}(O)|_{S} = H^{\infty}(U)|_{S}$.

Corollary 3.1.

If there exists compact subsets X_n, $X_n \subset X_{n+1}, \dots$ of X such that $R(X_n) = A(X_n)$ and $X\setminus\overline{S} \subset \bigcup_n X_n$ and if $A(X)$ is pointwise boundedly dense in $H^{\infty}(U)$. Then the conclusions of theorem 3.1 holds.

Proof of the corollary.

The hypothesis of the corollary implies via Vitushkin's theorem (theorem 8.2 in [4]) and theorem 2.2 of [5] that $(*)$ holds.

The proof of theorem 3.1 starts with the following lemma:

Lemma 3.1.

Assume the hypothesis of theorem 3.1.

Suppose $K \subset (\partial X)\setminus(\overline{S}\setminus S)$ is compact and $V \supset K$ is open. Let $\varepsilon > 0$.

There exists an open set $V_o \supset K$ and a constant M such that if $h|_U \in H^\infty(U)$ and $\|h\|_\infty = 1$ we can find a bounded function f on $\mathbb{C}$ analytic in $X^o \cup V_o$ such that $\|f-h\|_{\mathbb{C}\setminus V} < \varepsilon$ and $\|f\| \leq M\|h\|_V$ where M depends only on c and r.

Proof.

From the hypothesis we have

(*) $\gamma(\Delta(z,\delta) \setminus U) \leq c\, \gamma((\Delta(z,r\ ) \setminus X)$

Suppose ϕ is continuous differentiable and supported on $\Delta = (z,\delta)$. Then by (*):

(I) $|T\,\phi h'(\infty)| = |\frac{1}{\pi} \iint h \frac{\partial\phi}{\partial \bar z} dxdy| \leq 4\delta c\|h\|_\Delta \|\frac{\partial\phi}{\partial \bar z}\| \gamma(\Delta(z,r\delta) \setminus X)$

Now we use some of the notation from section 2. Put $G_k = T_{\phi_k} h$ where $\phi_{k\delta} \in C_0^1(\Delta(z_k,\delta))$ and let $E_k = \Delta(z,r\delta) \setminus X$. Then by (I) we have if $\Delta_k = \Delta(z_k,\delta)$

(II) $|G_k'(\infty)| \leq 16c\,\|h\|_{\Delta_k} \gamma(E_k)$.

Let W_k be the analytic center of E_k and $\beta(E_k)$ the analytic diameter of E_k.

Let $I = \{k\colon \Delta_k \cap K \neq \phi\}$. We can assume $V \cap \bar S = \phi$ and δ chosen so small that $\Delta(z_k\ (r+2)\delta) \subset V$ if $k \in I$. Then it follows from the proof of iii) => i) in Theorem 8.1 in [4] that

(III) $\left|\beta(G_k\ W_k)\right| = \left|\frac{1}{\pi}\iint h(z-W_k)\frac{\partial\phi_k}{\partial \bar z}\right| \leq c\cdot k(r)\|h\|_V\ \gamma(E_k)\beta(E_k)$

where $k(r)$ depends only on r.

Now it follows from lemma 6.3 in [ ] that there exsist functions $f_1^{(k)}$, $f_2^{(k)}$ analytic outside a compact subset of E_k such that

$\|f^{(k)}\| + \|f^{(k)}\| \leq 20$ and such that $0 = f_1(\infty) = f_2(\infty) = f_1'(\infty) = \beta(f_2, W_k)$ and $f_2'(\infty) = \gamma(E_k)$, $\beta(f_1, W_k) = \gamma(E_k)\cdot\beta(E_k)$.

But then we can choose complex numbers a and b such that $H_k = a\, f_1^k + b\, f_2^k$ satisfies

i) $G_k - H_k$ has a triple zero at infinity

ii) $\|H\|_k \leq c\, M(r)\|h\|_V$ where $M(r)$ depends only on r.

It is important that the singularities of H_k depends only on the singularities of f_1^k and f_2^k.

Define now $f = h - \sum_I (G_k - H_k)$. We have $\|G_k - H_k\| \leq a\|h\|_V$ where a depends only on r and c.

Since $\mathrm{dist}(K\, \mathbf{C} \setminus V) > 0$ we can exactly as at p.193 in |5| show that $\|f - h\|_{\mathbb{C}\setminus V} < \varepsilon$ if δ is small. It is important that we can use the same δ for any h satisfying $\|h\|_\infty \leq 1$.

Since $\sum_I \phi_k \equiv 1$ in a neighbourhood of K the function

$$h - \sum_I G_k = h - T_{\sum_I \phi_k}(h)$$ is analytic in U and in a neighbourhood V' of K. (V' depends only on δ). Remembering how the functions H_k were chosen we have proved the lemma.

The proof of the next lemma is almost a copy of an argument from section 2.

<u>Lemma 3.2.</u>

Suppose $\varepsilon > 0$ and assume the hypothesis of theorem 3.1.

There exists a constant k and an open set $0 \supset X\setminus(\overline{S}\setminus S)$ such that if $h \in H^\infty(U)$ and $\|h\| \leq 1$, there exsist $f \in H^\infty(0)$ such that $\|f\| \leq k$ and $\|f - h\|_S < \varepsilon$.

Proof:

We put $\partial X \setminus (\overline{S} \setminus S) = \bigcup_1^\infty K_n$ where each K_n is compact and $V_n \supset K_n$ is open such that $V_n \cap V_m \neq \phi \Rightarrow |n-m| \leq 1$, and $V_n \cap \overline{S} = \phi \ \forall n$ and $V_n \cap K \neq \phi$ only for finitely many n if $K \subset \mathbb{C} \setminus (\overline{S} \setminus S)$ is compact.

Looking at the functions H_k constructed in lemma 3.1 and noting that for a general bounded measurable function f $T_{\phi_k} f$ is analytic wherever f is analytic, we see that the technique used in the proof of theorem 2.1 combined with lemma 3.1 yields a function f such that

i) $\|f\|_\infty \leq 2M+2 \overset{\text{def}}{=} k$ (M is as in lemma 3.1)

ii) $\|f-h\|_S < \varepsilon$

iii) f is analytic in U and in an open set containing $\bigcup_1^\infty K_n$ and this open set does not depend on h.

We now prove theorem 3.1.

Since $H^\infty(U)|_S$ is closed in $C(S)$ there exists a constant L such that every $g \in C(S)$ equals $h|_S$ where $h \in H^\infty(U)$ and $\|h\| \leq L\,\|g\|$.
Let $\varepsilon > 0$.

We choose the open set O as in lemma 3.2 and apply the lemma to $\frac{h}{L}$.
We get a function $f_1 \in H^\infty(O)$ such that $\|f_1\| \leq k$ (k is as in lemma 8).

Then the function $f = Lf_1$ satisfies

i) $\|g-f\|_S \leq \varepsilon L$

ii) $\|f\| \leq k\,L$

If we choose $\varepsilon < \frac{1}{2L}$ we can prove theorem 3.1 via theorem 1.4.

Final Remarks.

We would like to comment theorem 3.1 a little.

Suppose A is a closed subspace of $H^{\infty}(U)$ and $T\phi\bar{h}|_U \in A$ whenever $\bar{h}$ is a bounded measurable extension of an $h \in A$ and ϕ is continuously differentiable with compact support. We shall then say that A is invariant under $T\phi$. Then the following result holds :

Corollary 3.1.

Suppose $A \subset H^{\infty}(U)$ is invariant under $T\phi$ and $h|_U \in A$ whenever h is analytic in a neighbourhood of X. If it is possible to choose the sets $\{V_n\}$ appearing in the proof of theorem 3.1 in such a way that $f \in A$ whenever $f = \lim_n f_n$ where $\{f_n\}$ is a bounded sequence from A and the convergence is uniform on those relatively closed subsets F of U satisfying $F \cap V_n \neq \phi$ only for finitely many n, then theorem 4 is valid with $H^{\infty}(U)$ replaced by A and $H^{\infty}(O)$ replaced by $H^{\infty}(O) \cap A$.

Example of such an A :

Let $U = \{z: |z| < 1\}$ and let $Q \subset \partial U$. Define A as those $f \in H$ such that $\lim_{r \to 1} f(r e^{i\theta})$ exsist whenever $e^{i\theta} \in Q$.

At last we wish to point out that if the diameters of the components of the complement of X is bounded away from zero, and explicit contruction of the set O in theorem 3.1 can be carried out. This depends on some estimates of the analytic capacity and diameter of compact connected sets. (See theorem 2.1 and lemma 6.1 in Ch. VIII of [4]).

REFERENCES

|1| Akutowicz & Carleson.
Journal d'Analyse mathematique. 1959-1960. 223-247.

|2| Curtis & Figa - Talamanca
Symposium on Function Algebras. Scoll Foresman and Co. 1966. 169-185.

|3| J. Detraz
Comptes Rendus Acad. Sc. Paris 269 serie A. 688-691.

|4| Gamelin: Uniform Algebras
Prentice Hall, Englewood Cliffs. N.J. 1969.

|5| Gamelin & Garnett.
Trans.Am.Math.Soc.

|6| Heard & Wells
Pacific J. Math. 28, 1969. 543-553.

|7| Hoffmann: Banach Spaces of Analytic Functions.
Prentice Hall, Englewood Cliffs. N.J. 1962.

|8| Vitushkin.
Russian Math. Surveys 22 (1968), 139-200.